Kuwait University	**Math 101**	**Date:**	**August 15, 2009**
Dept. of Math. & Comp. Sci.	**Final Exam**	**Duration:**	**Two Hours**

Calculators, cellular phones and all other mobile communication equipments are not allowed

Answer the following questions. Each question weighs 4 points.

1. Evaluate the following limits, if they exist:

   (a) $\lim_{x\to 0} \frac{\sec 3x \ \tan 3x}{5x}$ (b) $\lim_{x\to -\infty} \frac{\sqrt{2x^2+1}}{x+1}$

2. Find $f'(x)$, where $f(x) = \frac{x\cot x^2}{x^5+1}$.

3. Let $f(x) = 2x + |x|$.

   (a) Find the points, if any, where f is discontinuous. Justify your answer.

   (b) Find the points, if any, where f is not differentiable. Justify your answer.

4. Find equations of the lines passing through the origin and tangent to the curve
$$y = x^2 + 1.$$

5. Let x and y be two positive numbers whose sum is 4. Find the values of x and y that minimize the function $P = x^2 + y^2$.

6. Prove that: $\int_1^4 \cos^2 x \, dx = 3 - \int_1^4 \sin^2 x \, dx$

7. Let $f(x) = \int_0^{x^2} \frac{t-4}{t^2+1} \, dt$. Find the intervals on which f is increasing and the intervals on which f is decreasing.

8. Evaluate:

   (a) $\int_0^3 \frac{x}{\sqrt{16+x^2}} \, dx$, (b) $\int \frac{(\tan x + 7)^7}{\cos^2 x} \, dx$

9. Find the area of the region bounded by the curves $y = x + 2$ and $x = y^2 - 4$.

10. The region bounded by the curves $y = x^2$ and $y = x + 2$ is revolved about:

    (a) the line $y = -1$,

    (b) the line $x = 3$.

    Set up an integral that can be used to find the volume of the resulting solid in each case.

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1. (a) $\lim\limits_{x\to 0}\frac{\sec 3x\ \tan 3x}{5x}=\frac{3}{5}\left(\lim\limits_{x\to 0}\sec 3x\right)\times\left(\lim\limits_{x\to 0}\frac{\tan 3x}{3x}\right)=\boxed{\frac{3}{5}}$ $[\lim\limits_{x\to 0}3x=0]$.

   (b) $\lim\limits_{x\to-\infty}\frac{\sqrt{2x^2+1}}{x+1}=\lim\limits_{x\to-\infty}\frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{x\left(1+\frac{1}{x}\right)}=\lim\limits_{x\to-\infty}\frac{|x|\ \sqrt{2+\frac{1}{x^2}}}{x\left(1+\frac{1}{x}\right)}=\lim\limits_{x\to-\infty}\frac{-x\ \sqrt{2+\frac{1}{x^2}}}{x\left(1+\frac{1}{x}\right)}=\boxed{-\sqrt{2}}$

2. $f'(x)=\frac{\left(x^5+1\right)\left[\cot x^2-2x^2\csc^2 x^2\right]-5x^5\cot x^2}{\left(x^5+1\right)^2}$

3. (a) $2x$ (polynomial function) and $|x|$ are continuous therefore, $f(x)=2x+|x|$ is continuous, [their sum].

   (b) $R.H.D.$ at $(x=0)$ $\lim\limits_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0^+}\frac{[2x+x]-0}{x}=3$, $L.H.D.$ at $(x=0)=\lim\limits_{x\to 0^-}\frac{[2x-x]-0}{x}=1$, f is continuous at $x=0$. f has a corner at $x=0$ therefore, f is not differentiable at $x=0$.

4. $y'=2x$. Let the point of tangency be $P\left(a,a^2+1\right)$ {slope of tangent at $P=2a$}. Equation of tangent line at $P: y-\left(a^2+1\right)=2a\left(x-a\right)$. Since the tangent passes by the origin, then $0-\left(a^2+1\right)=2a\left(0-a\right)\implies a^2=1\implies a=\pm 1$. Equation of tangent lines are: $\boxed{y-2=2\left(x-1\right)\ \&\ y-2=-2\left(x+1\right)}$. OR $\boxed{y=2x}$ & $\boxed{y=-2x}$.

5. $x+y=4\implies S=x^2+\left(4-x\right)^2,\ x\in(0,4)\implies\boxed{\frac{dS}{dx}=4x-8}$. If $\frac{dS}{dx}=0$, then $x=2$ is critical number. $\left.\frac{d^2S}{dx^2}\right|_{x=2}>0$. S is minimum at $x=2$ and $y=2$.

6. $\int\limits_1^4\cos^2 x\ dx+\int\limits_1^4\sin^2 x\ dx=\int\limits_1^4\left(\cos^2 x+\sin^2 x\right)dx=\int\limits_1^4 dx=3\implies\int\limits_1^4\cos^2 x\ dx=3-\int\limits_1^4\sin^2 x\ dx.$

7. $f'(x)=\frac{d}{dx}\int\limits_0^{x^2}\frac{t-4}{t^2+1}\ dt=\frac{x^2-4}{x^4+1}(2x)=\frac{2x\left(x-2\right)\left(x+2\right)}{x^4+1}\implies$ If $f'(x)=0$, then $\boxed{x=\pm 1,0}$ are critical numbers

I	$(-\infty,-2)$	$(-2,0)$	$(0,2)$	$(2,\infty)$
sign of $f'(x)$	$-$	$+$	$-$	$+$
Conclusion	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

f is increasing on $[-2,0]\cup[2,\infty)$ and decreasing on $(-\infty,-2]\cup[0,2]$.

8. (a) Put $u=16+x^2\implies u(0)=16, u(3)=25\&du=2xdx\implies\int\limits_0^3\frac{x}{\sqrt{16+x^2}}dx=\frac{1}{2}\int\limits_{16}^{25}\frac{du}{\sqrt{u}}=\frac{1}{2}\left[2u^{\frac{1}{2}}\right]_{16}^{25}=\boxed{1}$

   (b) Put $u=\tan x+7\implies du=\sec^2 xdx=\frac{dx}{\cos^2 x}\implies\int\frac{\left(\tan x+7\right)^7}{\cos^2 x}dx=\int u^7du=\frac{u^8}{8}+C=\boxed{\frac{1}{8}\left(\tan x+7\right)^8+C}$.

9. A is an R_y-region. Points of intersection $(0,2)$ & $(-3,-1)$ $\Longrightarrow$ $Area = \int_{-1}^{2} \left[(y-2)-(y^2-4)\right] dy =$

$\left[-\frac{y^3}{3}+\frac{y^2}{2}+2y\right]_{-1}^{2} = \boxed{\frac{9}{2}}$

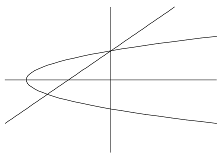
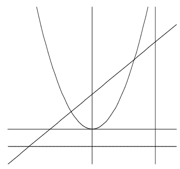
10. The region is an R_x-region.

(a) Revolution about the line $y=-1$:

$$Volume = \pi \int_{-1}^{2} \left\{ \left[(x+2)-(-1)\right]^2 - \left[x^2-(-1)\right]^2 \right\} dx = \pi \int_{-1}^{2} \left[(x+3)^2 - (x^2+1)^2 \right] dx$$

(b) Revolution about the line $x=3$:

$$Volume = 2\pi \int_{-1}^{2} (3-x)\left[(x+2)-(x^2)\right] dx = 2\pi \int_{-1}^{2} (3-x)\left(-x^2+x+2\right) dx$$

Q9:  , Q10:  .